Fundamental Principles of Digital Signal Processing

Lantao Yu, *Student Member, IEEE*

I. Notations

- Filter: $h[n]$. $h[n]$ can be real-valued or complex-valued. $h[n]$ is a N-tap filter. $0 \leqslant n \leqslant N-1$.
- DTFT $\{h[n]\} = H(\omega)$. DFT $\{h[n]\} = H(k)$.
- Real-valued signal: $x[n]$. The length of $x[n]$ is L. DTFT $\{x[n]\} = X(\omega)$. DFT $\{x[n]\} = X(k)$.
- Complex-valued signal: $y[n]$. The length of $y[n]$ is L. DTFT $\{y[n]\} = Y(\omega)$. DFT $\{y[n]\} = Y(k)$.
- Circular shift: $y[((n-m))_L]$ is a sequence that circularly shifts $y[n]$ to the right by m units.
- Circular convolution:
$$z[n] = y[n] \circledast h[n] = \sum_{m=0}^{N-1} y[n]h[((n-m))_N].$$
- We assume the input signals are one-dimensional. These signals are generic in a sense that their frequency responses are non-zero at any frequency in the range from $-\pi$ to π.

II. Principles and Proofs

1. Convolving an even-length filter with an even-length filter gives rise to an odd-length filter; convolving an even-length filter with an odd-length filter gives rise to an even-length filter; convolving an odd-length filter with an odd-length filter gives rise to an odd-length filter.

   Proof: The convolution of a N_1-tap filter with the other N_2-tap filter gives rise to a new filter of length $N_1 + N_2 - 1$. When N_1 and N_2 are both even, $N_1 + N_2 - 1$ is odd. When N_1 is even and N_2 is odd, $N_1 + N_2 - 1$ is even. When N_1 and N_2 are both odd, $N_1 + N_2 - 1$ is odd.
2. Convolving any integer number of odd-length filters gives rise to an odd-length filter.

   Proof: Assume we have l N-tap filters for convolution, the output is a filter with the length of $Nl-(l-1) = (N-1)l-1$. Since $N-1$ is even, $(N-1)l$ is also even. Therefore, $(N-1)l-1$ is odd.
3. Convolving any even number of even-length filters gives rise to an odd-length filter.

   Proof: Assume we have l N-tap filters for convolution, the output is a filter with the length of $Nl-(l-1) = (N-1)l-1$. Since $N-1$ is odd and l is even, $(N-1)l$ is also even. Therefore, $(N-1)l-1$ is odd.

4. Convolving any odd number of even-length filters gives rise to an even-length filter.

   Proof: Assume we have l N-tap filters for convolution, the output is a filter with the length of $Nl-(l-1) = (N-1)l-1$. Since $N-1$ is odd and l is odd, $(N-1)l$ is also odd. Therefore, $(N-1)l-1$ is even.

5. $X(-\epsilon) = X^*(\epsilon)$ and $X(\pi-\epsilon) = X^*(\pi+\epsilon)$.

   Proof: Since

$$X(\epsilon) = \sum_{n=0}^{N-1} x[n]e^{-jn\epsilon},$$

   we have

$$\begin{aligned} X(-\epsilon) &= \sum_{n=0}^{N-1} x[n]e^{jn\epsilon} \\ &= \left(\sum_{n=0}^{N-1} x^*[n]e^{-jn\epsilon}\right)^* \\ &= \left(\sum_{n=0}^{N-1} x[n]e^{-jn\epsilon}\right)^* \\ &= X^*(\epsilon). \end{aligned}$$

   We also have

$$\begin{aligned} X(\pi-\epsilon) &= \sum_{n=0}^{N-1} x[n]e^{-jn(\pi-\epsilon)} \\ &= \left(\sum_{n=0}^{N-1} x^*[n]e^{jn(\pi-\epsilon)}\right)^* \\ &= \left(\sum_{n=0}^{N-1} x^*[n]e^{jn(-\pi-\epsilon)}\right)^* \\ &= \left(\sum_{n=0}^{N-1} x^*[n]e^{-jn(\pi+\epsilon)}\right)^* \\ &= \left(\sum_{n=0}^{N-1} x[n]e^{-jn(\pi+\epsilon)}\right)^* \\ &= X^*(\pi+\epsilon). \end{aligned}$$

6. DTFT $\{h^*[n]\} = H^*(-\omega)$.

   Proof:

   Since

$$H(\omega) = \sum_{n=0}^{N-1} h[n]e^{-jn\omega},$$

we have

$$\begin{aligned}\text{DTFT}\left\{h^*[n]\right\} &= \sum_{n=0}^{N-1} h^*[n]e^{-jn\omega} \\ &= \left(\sum_{n=0}^{N-1} h[n]e^{jn\omega}\right)^* \\ &= \Big(H(-\omega)\Big)^* \\ &= H^*(-\omega).\end{aligned}$$

7. $\text{DTFT}\left\{y[n-m]\right\} = e^{-jm\omega}Y(\omega)$.

   Proof:

   Since

$$Y(\omega) = \sum_{n=0}^{N-1} y[n]e^{-jn\omega},$$

   we have

$$\begin{aligned}\text{DTFT}\left\{y[n-m]\right\} &= \sum_{n=0}^{N-1} y[n-m]e^{-jn\omega} \\ &= \sum_{k=0}^{N-1} y[k]e^{-j(m+k)\omega} \\ &= \left(\sum_{k=0}^{N-1} y[k]e^{-jk\omega}\right)e^{-jm\omega} \\ &= e^{-jm\omega}Y(\omega).\end{aligned}$$

8. $\text{DFT}\left\{y[((n-m))_L]\right\} = e^{-j\frac{2\pi}{L}mk}Y(k)$.

   Proof:

   Since

$$\text{DFT}\left\{y[n]\right\} = Y(k) = \sum_{n=0}^{N-1} y[n]e^{-j\frac{2\pi}{L}nk},$$

   we have

$$\begin{aligned}
\text{DFT}\left\{y[((n-m))_L]\right\} &= \sum_{n=0}^{N-1} y[((n-m))_L]e^{-j\frac{2\pi}{L}nk} \\
&= \sum_{n=0}^{m-1} y[L-m+n]e^{-j\frac{2\pi}{L}nk} + \sum_{n=m}^{L-1} y[n-m]e^{-j\frac{2\pi}{L}nk} \\
&= \sum_{n=0}^{m-1} y[L-m+n]e^{-j\frac{2\pi}{L}(n+L)k} + \sum_{n=m}^{L-1} y[n-m]e^{-j\frac{2\pi}{L}nk} \\
&= \sum_{n=L}^{L+m-1} y[n-m]e^{-j\frac{2\pi}{L}nk} + \sum_{n=m}^{L-1} y[n-m]e^{-j\frac{2\pi}{L}nk} \\
&= \sum_{n=m}^{L+m-1} y[n-m]e^{-j\frac{2\pi}{L}nk} \\
&= \sum_{n=m}^{L+m-1} y[n-m]e^{-j\frac{2\pi}{L}(n-m)k}e^{-j\frac{2\pi}{L}mk} \\
&= e^{-j\frac{2\pi}{L}mk}Y(k)
\end{aligned}$$

9. When $h[n] = h^*[N-1-n]$, for $n = 0, \dots, N-1$, $H^*(\omega) = H(\omega)e^{j\omega(N-1)}$.

   Proof:

   Since
$$H(\omega) = \sum_{n=0}^{N-1} h[n]e^{-jn\omega},$$
   we have
$$\begin{aligned}
H^*(\omega) &= \sum_{n=0}^{N-1} h^*[n]e^{jn\omega} \\
&= \sum_{k=0}^{N-1} h^*[N-1-k]e^{j(N-1-k)\omega} \\
&= \sum_{k=0}^{N-1} h[k]e^{j(N-1-k)\omega} \\
&= \sum_{k=0}^{N-1} h[k]e^{-jk\omega}e^{j(N-1)\omega} \\
&= H(\omega)e^{j(N-1)\omega}.
\end{aligned}$$

10. DFT $\{z[((n+N-1))_L]\} = H^*(k)Y(k)$, when $z[n] = y[n] \circledast h[n]$ and $h[n] = h^*[N-1-n]$.

    Proof:

    Since $z[n] = y[n] \circledast h[n]$, we have $Z[k] = H[k]Y[k]$. Since $H^*(\omega) = H(\omega)e^{j\omega(N-1)}$, we have $H^*[k] = H[k]e^{j\frac{2\pi}{N}(N-1)}$. Therefore, $H^*(k)Y(k) = e^{j\frac{2\pi}{N}(N-1)}H[k]Y[k]$. Since DFT $\{y[((n-m))_L]\} = e^{-j\frac{2\pi}{L}mk}Y(k)$, we therefore have:

$$\begin{aligned}
\text{DFT}\left\{z[((n+N-1))_L]\right\} &= e^{j\frac{2\pi}{N}(N-1)}Z[k] \\
&= e^{j\frac{2\pi}{N}(N-1)}H[k]Y[k] \\
&= H^*(k)Y(k).
\end{aligned}$$

11. The N-tap, real-valued filter $h[n]$ has its DTFT with linear-phase response when N is even and $h[n] = h[N-1-n]$, for $0 \leqslant n \leqslant N-1$.

Proof:

$$\begin{aligned}
H(\omega) &= \sum_{n=0}^{N-1} h[n]e^{-jn\omega} \\
&= \sum_{n=0}^{\frac{N}{2}-1} h[n]e^{-jn\omega} + \sum_{n=\frac{N}{2}}^{N-1} h[n]e^{-jn\omega} \\
&= \sum_{n=0}^{\frac{N}{2}-1} h[n]e^{-jn\omega} + \sum_{n=0}^{\frac{N}{2}-1} h[N-1-n]e^{-j(N-1-n)\omega} \\
&= \sum_{n=0}^{\frac{N}{2}-1} h[n]e^{-jn\omega} + \sum_{n=0}^{\frac{N}{2}-1} h[N-1-n]e^{-j(N-1)\omega}e^{-jn\omega} \\
&= e^{-j\frac{N-1}{2}\omega} \sum_{n=0}^{\frac{N}{2}-1} h[n]\left(e^{-jn\omega}e^{j\frac{N-1}{2}\omega} + e^{jn\omega}e^{-j\frac{N-1}{2}\omega}\right) \\
&= e^{-j\frac{N-1}{2}\omega} \sum_{n=0}^{\frac{N}{2}-1} 2h[n]\cos\left[\left(\frac{N-1}{2} - n\right)\omega\right] \\
&= e^{-j\frac{N-1}{2}\omega} \sum_{n=1}^{\frac{N}{2}} 2h\left[\frac{N}{2} - n\right]\cos\left[\left(n - \frac{1}{2}\right)\omega\right] \\
&= e^{-j\frac{N-1}{2}\omega} H_r(\omega),
\end{aligned}$$

where

$$H_r(\omega) = \sum_{n=1}^{\frac{N}{2}} 2h\left[\frac{N}{2} - n\right]\cos\left[\left(n - \frac{1}{2}\right)\omega\right]$$

is real-valued. When ω covers the main lobe of the filter response, $H_r(\omega)$ will always be positive or negative in the ranage of ω.